Infinite Limits and Asymptotes

Math 130 - Essentials of Calculus

15 February 2021

Infinite Limits

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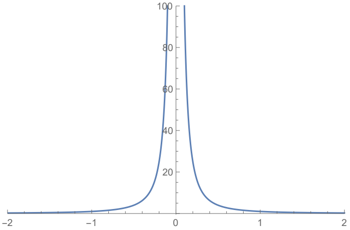

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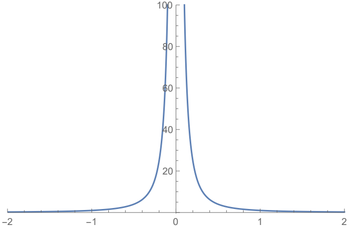

we see that the function is getting arbitrarily large from both sides as we approach 0. Thus we say that

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

Infinite Limits

Definition

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Infinite limits happen when vertical asymptotes occur, and so this actually gives us a second definition:

Definition

The line $x = a$ *is called a vertical asymptote of the curve* $y = f(x)$ *if at least one of the following statements is true:*

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

Examples

Example

Compute the following limits

1. $\displaystyle\lim_{x\to -3^+} \frac{x+2}{x+3}$

2. $\displaystyle\lim_{x\to -3^-} \frac{x+2}{x+3}$

3. $\displaystyle\lim_{x\to -3} \frac{x+2}{x+3}$

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1. $\displaystyle\lim_{x\to-3^+}\frac{x+2}{x+3}$
2. $\displaystyle\lim_{x\to-3^-}\frac{x+2}{x+3}$
3. $\displaystyle\lim_{x\to-3}\frac{x+2}{x+3}$
4. $\displaystyle\lim_{x\to2^-}\frac{1-x}{(x-2)^2}$
5. $\displaystyle\lim_{x\to2^+}\frac{1-x}{(x-2)^2}$
6. $\displaystyle\lim_{x\to2}\frac{1-x}{(x-2)^2}$

Limits at Infinity

Let's now look at the graph of the function $f(x) = \dfrac{x^2 - 1}{x^2 + 1}$ and study its values as x gets arbitrarily large.

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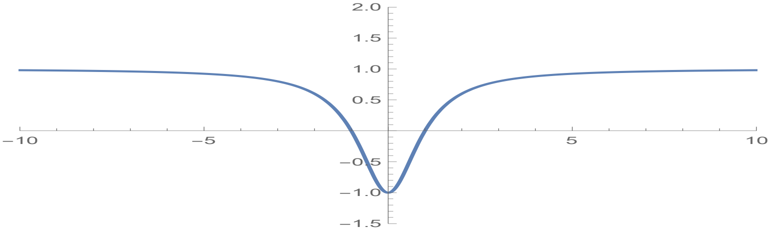
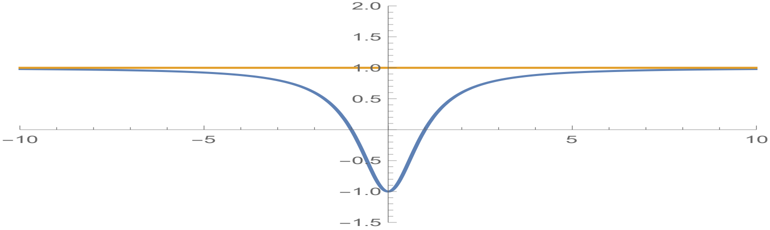


What does it look like the values of this function are approaching as $x \to \pm\infty$?

Limits at Infinity



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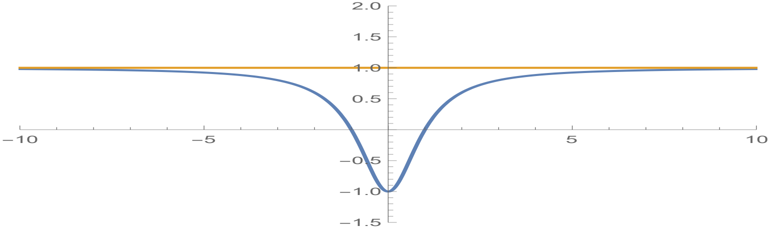

Based on this, we could say $\lim_{x \to \infty} \dfrac{x^2 - 1}{x^2 + 1} = 1$

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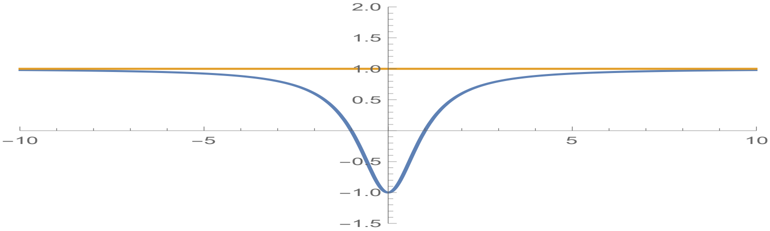
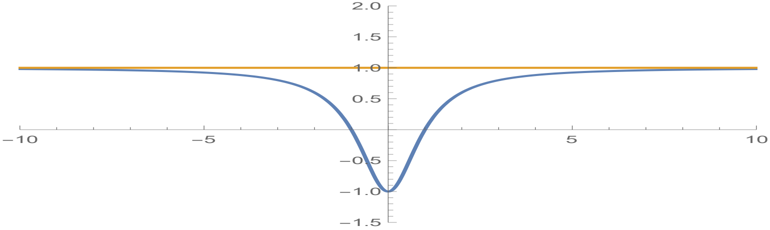


Based on this, we could say $\lim_{x\to\infty} \dfrac{x^2-1}{x^2+1} = 1$ as well as $\lim_{x\to-\infty} \dfrac{x^2-1}{x^2+1} = 1$.

Limits at Infinity



Based on this, we could say $\lim_{x\to\infty} \frac{x^2-1}{x^2+1} = 1$ as well as $\lim_{x\to-\infty} \frac{x^2-1}{x^2+1} = 1$. Moreover, we can see that f has a horizontal asymptote at $y = 1$.

Limits at Infinity

Definition

Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ approach L as x becomes large.

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Limits at infinity also detect horizontal asymptotes for us:

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Limits at infinity also detect horizontal asymptotes for us:

Definition

The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L.$$

Basics of Limits at Infinity

For a fraction $\dfrac{a}{b}$, the larger the number b is, the smaller the value of the overall fraction is.

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Using these limits, we can compute many limits at infinity.

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Compute the limits

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1. $$\lim_{x \to \infty} \frac{x^3 + 5x}{2x^3 - x^2 + 4}$$
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1. $$\lim_{x \to \infty} \frac{x^3 + 5x}{2x^3 - x^2 + 4}$$
2. $$\lim_{x \to \infty} \frac{2x^2 - 1}{4x^2 + x}$$
3. $$\lim_{x \to -\infty} \frac{t^2 + 2}{t^3 + t^2 - 1}$$

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2. $$\lim_{x\to\infty} \frac{2x^2-1}{4x^2+x}$$
3. $$\lim_{x\to-\infty} \frac{t^2+2}{t^3+t^2-1}$$
4. $$\lim_{x\to\infty} \frac{x+x^3+x^5}{1-x^2+x^4}$$

Final Example

Example

Find all vertical and horizontal asymptotes of the curve

$$y = \frac{2x^2 + x - 1}{x^2 + x - 2}$$